J. Fluid Mech. (1973), *vol.* 58, *part* 2, *pp.* 337–351
Printed in Great Britain

Subcritical bifurcation of plane Poiseuille flow

By T. S. CHEN

Department of Mechanical and Aerospace Engineering, University of Missouri-Rolla

AND D. D. JOSEPH

Department of Aerospace Engineering and Mechanics,
University of Minnesota

(Received 25 October 1972)

We apply the perturbation theory which was recently developed and justified by Joseph & Sattinger (1972) to determine the form of the time-periodic solutions which bifurcate from plane Poiseuille flow. The results at lowest significant order seem to be in good agreement with those following from the formal perturbation method of Stuart (1960) as extended by Reynolds & Potter (1967). Given the numerical results of the present calculation, the rigorous theory guarantees that the only time-periodic solution which bifurcates from laminar Poiseuille flow is a two-dimensional wave. The wave which bifurcates at the lowest Reynolds number exists, but it is unstable when its amplitude is small. Solutions which escape the small domain of attraction of laminar Poiseuille flow snap through this unstable time-periodic solution with a small amplitude to solutions of larger amplitudes.

1. Introduction

We can study stability of steady solutions of the Navier–Stokes equations by considering how the eigenvalues $\gamma(R) = \xi(R)+i\eta(R)$ of the spectral problem of the linearized theory of stability vary with the Reynolds number R. Joseph & Sattinger (1972, hereafter called JS) have shown that, if the principal eigenvalue at criticality ($\gamma(R_c) = i\omega_0$) is simple and $\omega_0 \neq 0$, then a finite amplitude, time-periodic solution bifurcates from the steady solution. The time-periodic solutions† exist only when the frequency $\omega(\epsilon)$ and Reynolds number $R(\epsilon)$ have certain values which depend on the amplitude ϵ. The solution and the values $R(\epsilon)$ and $\omega(\epsilon)$ can be obtained as perturbation series in powers of ϵ which converge and solve the problem (JS). The implicit function theorem which is used to prove convergence of the series solution also guarantees that a unique analytic branch passes through the critical point $(R(\epsilon), \epsilon) = (R_c, 0)$ of the bifurcation diagram.

Unlike the problem of bifurcation of steady solutions from steady solutions in which two-sided bifurcations are possible, the time-periodic problem will

† These solutions are unique to within an arbitrary phase. This degree of arbitrariness follows from the invariance of the bifurcation problem to changes in the origin of time.

allow either supercritical branching ($R(\epsilon)-R_c > 0$) or subcritical branching ($R(\epsilon)-R_c < 0$), but not both.† In the steady case,

$$R(\epsilon)-R_c = R_1\epsilon+R_2\epsilon^2+\dots \tag{1.1}$$

and, in general, $R_1 \neq 0$ (see Joseph 1971). Therefore, $R-R_c$ changes sign as ϵ passes through zero. In the time-periodic case $R = R(\epsilon^2)$ and $\omega = \omega(\epsilon^2)$, and $R(\epsilon^2)-R_c$ cannot change sign as ϵ passes through zero.

The bifurcating solutions have an important property of stability when ϵ is small and $\gamma(R_c) = i\omega_0$ is a simple eigenvalue of the spectral problems.

(*a*) Supercritical bifurcating solutions are stable to small disturbances.

(*b*) Subcritical bifurcating solutions are unstable to small disturbances.

These results (see JS) have a central place in the theory of hydrodynamic stability. In case (*a*) the exchange of stability between the steady solution and the bifurcating solution is a continuous process involving the smooth development of new modes of motion. In the second case (*b*) the bifurcating solution is unstable; disturbances which escape the domain of attraction of the steady solution snap through the unstable bifurcating solution and are attracted to (possibly 'turbulent') solutions with larger norms.

The present work is essentially an application of the perturbation method developed and justified in JS to the bifurcating problem for plane Poiseuille flow. This bifurcating problem has been treated, using a formal method of amplitude expansion, by Stuart (1960), Watson (1960), Reynolds & Potter (1967), Pekeris & Shkoller (1967, 1969) and McIntire & Lin (1972). The results of our analysis at lowest order are not all directly comparable with the results of these other authors because they calculated 'Landau constants' and we calculate derivatives of the analytic functions $R(\epsilon^2)$ and $\omega(\epsilon^2)$. However, at lowest significant order our results and those of the other authors mentioned do appear to be in good agreement where they can be compared.

To a considerable degree the outcome of the bifurcation analysis of plane Poiseuille flow is disappointing. Since the bifurcating solutions are subcritical when ϵ is small, they are unstable. Presumably disturbances are attracted to large norm solutions which are possibly beyond the reach of perturbation theory. It follows that the unstable time-periodic solution which we shall construct as a power series in ϵ will not represent the larger norm solutions which replace laminar Poiseuille flow in experiments.

To a certain extent, the present calculation gains more significance by virtue of its connexion with the general rigorous theory of JS. In particular, this connexion allows definite assertions about the uniqueness of the two-dimensional solution which bifurcates at criticality as well as a proof of the properties of the solutions which bifurcate from any point on the neutral curve for plane Poiseuille flow. We are also interested in demonstrating that the bifurcation theory of JS can be used for numerical calculations as well as for proofs about the qualitative properties of bifurcating solutions.

† We are assuming here that the steady solution *loses* stability when R is increased past R_c. When the steady solution *gains* stability as R is increased past R_c, subcritical solutions are those for which $R(\epsilon)-R_c > 0$. This is the situation on the upper branch of the neutral curve of figure 1.

2. The basic flow and the disturbance flow

In JS the bifurcation theory was developed for arbitrary steady solutions of the Navier–Stokes equations in a bounded domain.† Instead of specializing the results of JS for our plane Poiseuille flow problem, we are going to rederive the perturbation theory for two-dimensional disturbances using the stream function. The restriction to two-dimensional disturbances allows a more direct comparison with the method used by previous authors. Moreover, we shall argue in § 9 that the restriction to two-dimensional disturbances does not necessarily imply a loss in the generality of the results.

The Navier–Stokes equations for a viscous flow in a duct without body forces can be written in dimensionless form as

$$\frac{\partial}{\partial t}\tilde{\tilde{\mathbf{V}}}+\tilde{\tilde{\mathbf{V}}}.\nabla\tilde{\tilde{\mathbf{V}}}+\nabla\tilde{\tilde{p}}-\frac{1}{R}\Delta\tilde{\tilde{\mathbf{V}}}=0, \tag{2.1a}$$

with

$$\nabla.\tilde{\tilde{\mathbf{V}}}=0, \tag{2.1b}$$

$$\tilde{\tilde{\mathbf{V}}}=0 \quad \text{on the surface of domain } \Omega. \tag{2.1c}$$

In the equations, $\tilde{\tilde{\mathbf{V}}}=(\tilde{\tilde{U}},\tilde{\tilde{V}},\tilde{\tilde{W}})$ is the velocity, $\tilde{\tilde{p}}$ is the pressure, R is the Reynolds number and t is the time. Here we use the maximum velocity of the steady flow $\tilde{U}_{\max}$ and the half-height of the channel L as the characteristic velocity and length so that $R=\tilde{U}_{\max}L/\nu$, with ν denoting the kinematic viscosity.

For two-dimensional flow, the velocity field can be expressed in terms of a stream function $\tilde{\tilde{\Psi}}(x,y,t)$ as

$$\tilde{\tilde{\mathbf{V}}}=\mathbf{i}\frac{\partial\tilde{\tilde{\Psi}}}{\partial y}-\mathbf{j}\frac{\partial\tilde{\tilde{\Psi}}}{\partial x},\quad \tilde{\tilde{U}}=\frac{\partial\tilde{\tilde{\Psi}}}{\partial y},\quad \tilde{\tilde{V}}=-\frac{\partial\tilde{\tilde{\Psi}}}{\partial x}. \tag{2.2}$$

Substituting (2.2) into (2.1) and eliminating pressure terms yields

$$\frac{\partial}{\partial t}\Delta\tilde{\tilde{\Psi}}+\frac{\partial\tilde{\tilde{\Psi}}}{\partial y}\frac{\partial}{\partial x}\Delta\tilde{\tilde{\Psi}}-\frac{\partial\tilde{\tilde{\Psi}}}{\partial x}\frac{\partial}{\partial y}\Delta\tilde{\tilde{\Psi}}-\frac{1}{R}\Delta^2\tilde{\tilde{\Psi}}=0, \tag{2.3a}$$

$$\tilde{\tilde{\Psi}}=\partial\tilde{\tilde{\Psi}}/\partial y=0 \quad \text{at} \quad y=\pm 1. \tag{2.3b}$$

For the case of plane Poiseuille flow considered in the present analysis, the system (2.1) has a steady solution

$$\tilde{U}=1-y^2,\quad \tilde{V}=\tilde{W}=0. \tag{2.4}$$

It is known that the steady solution is globally stable when R is small. For large values of R, solution (2.4) may be stable to small disturbances and unstable to larger disturbances. At a yet larger value of $R=R_c$, solution (2.4) loses stability and is replaced by another type of stable solution, such as a secondary steady motion, or a time-periodic motion.

Let

$$\tilde{\tilde{\Psi}}=\tilde{\Psi}(y)+\epsilon\hat{\Psi}(x,y,t,R), \tag{2.5}$$

where $\tilde{\Psi}(y)=y-\frac{1}{3}y^3$ is the stream function for (2.4) and ϵ is an amplitude which will be more carefully defined later.

† The bifurcation of periodic solutions into quasi-periodic solutions has been treated by Joseph (1973).

Substitution of (2.5) into (2.3) leads to

$$\frac{\partial}{\partial t}\Delta\hat{\Psi}+\tilde{U}\frac{\partial}{\partial x}\Delta\hat{\Psi}-\tilde{U}''\frac{\partial\hat{\Psi}}{\partial x}-\frac{1}{R}\Delta^2\hat{\Psi}+\epsilon J(\hat{\Psi},\Delta\hat{\Psi})=0, \tag{2.6a}$$

$$\hat{\Psi}=\partial\hat{\Psi}/\partial y=0 \quad \text{at} \quad y=\pm 1, \tag{2.6b}$$

where $\hat{\Psi}$ is a periodic function of x with period $2\pi/\alpha$ (2.6c)

and

$$J(\hat{\Psi},\Delta\hat{\Psi})=\frac{\partial\hat{\Psi}}{\partial y}\frac{\partial}{\partial x}\Delta\hat{\Psi}-\frac{\partial\hat{\Psi}}{\partial x}\frac{\partial}{\partial y}\Delta\hat{\Psi}. \tag{2.7}$$

3. Instability of the basic flow

The spectral problem for the instability of (2.4) is obtained from (2.6) by setting $\epsilon = 0$ in (2.6a). We may then write (2.6) as

$$\left(\frac{\partial}{\partial t}\Delta+\mathscr{L}\right)\hat{\Psi}=0, \quad \hat{\Psi}=\frac{\partial\hat{\Psi}}{\partial y}=0 \quad \text{at} \quad y=\pm 1, \quad \text{periodicity in } x, \tag{3.1a, b, c}$$

where

$$\mathscr{L}=\mathscr{L}[\tilde{U},\lambda]=\tilde{U}\frac{\partial}{\partial x}\Delta-\tilde{U}''\frac{\partial}{\partial x}-\lambda\Delta^2 \tag{3.2}$$

and $\lambda = 1/R$.

To obtain the spectral problem, we seek solutions of (3.1) of the form

$$\hat{\Psi}(x,y,t,\lambda)=e^{-\gamma(\lambda)t}\psi(x,y,\lambda). \tag{3.3}$$

This leads to

$$-\gamma\Delta\psi+\mathscr{L}\psi=0, \quad \psi=\partial\psi/\partial y=0 \quad \text{at} \quad y=\pm 1, \quad \text{periodicity in } x. \tag{3.4a, b, c}$$

The values $\gamma(\lambda)=\xi(\lambda)+i\eta(\lambda)$ are eigenvalues of the spectral problem (3.4a, b, c). If $\xi(\lambda) < 0$ then plane Poiseuille flow $\tilde{U}$ is unstable.

For large values of λ (small R), $\xi(\lambda) > 0$ for all eigenvalues $\gamma(\lambda)$. The stability and instability border is defined by the critical values $\lambda=\lambda_0=1/R_c$, where $\xi(\lambda_0)=0$. At criticality $\gamma(\lambda_0)=i\eta(\lambda_0)=i\omega_0$ and $\mathscr{L}[\tilde{U},\lambda_0]=\mathscr{L}_0$.

Assume that $\gamma(\lambda)$ is a simple eigenvalue of (3.4). Then $\overline{\gamma}(\lambda)$ is also an eigenvalue, where $\overline{\gamma}=\xi-i\eta$ is the complex conjugate of γ. The functions

$$\psi(x,y,\lambda) \quad \text{and} \quad \overline{\psi}(x,y,\lambda) \tag{3.5}$$

are eigenfunctions of (3.4) belonging, respectively, to the eigenvalues γ and $\overline{\gamma}$ of (3.4).

A consequence of the requirement that the solutions (3.5) of (3.4) are periodic functions of x with period $2\pi/\alpha$ is that these functions may be expanded in Fourier series

$$\psi=\sum_n e^{i\alpha nx}\phi(n,y,\lambda). \tag{3.6}$$

Since (3.4) is a linear problem, each term of (3.6) must separately satisfy (3.4) and, without loss of generality, we may consider that all solutions are of the form

$$\psi(x,y,\lambda(\alpha))=e^{i\alpha x}\phi(y,\alpha,\lambda(\alpha)). \tag{3.7}$$

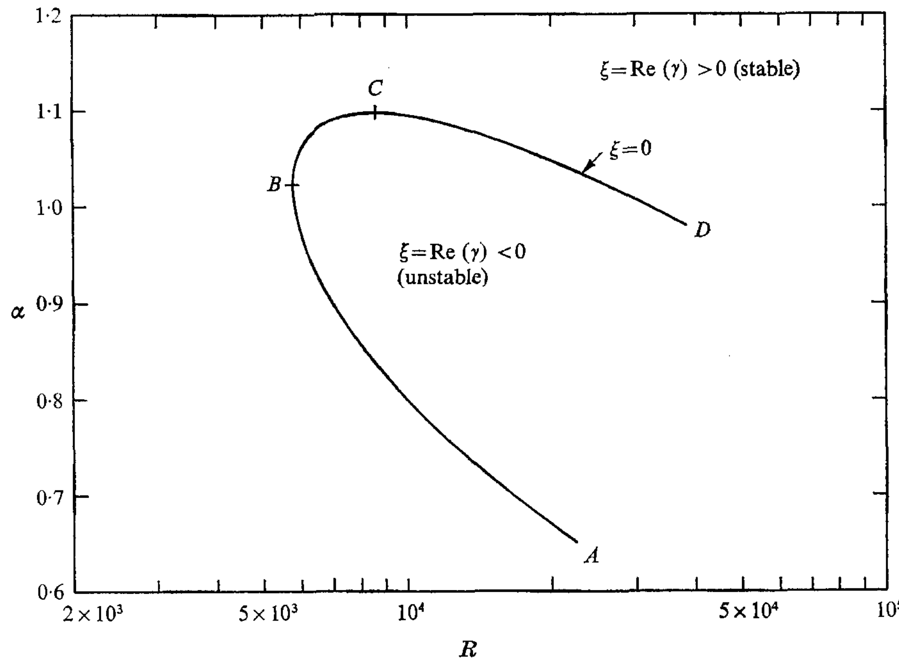

FIGURE 1. The neutral stability curve for plane Poiseuille flow.

Substitution of (3.7) into (3.4) leads to the Orr–Sommerfeld problem for plane Poiseuille flow.

The neutral curve is the locus of values $\lambda(\alpha)$ for which $\xi(\lambda(\alpha)) = 0$. This curve is shown in figure 1.

4. The adjoint problem

Define the scalar product

$$(a, b) = \int_\Omega a\bar{b}\, d\Omega = \int_0^{2\pi/\alpha} \int_{-1}^{1} a\bar{b}\, dy\, dx. \tag{4.1}$$

The operator $\mathscr{L}^*$ which is adjoint to $\mathscr{L}$ relative to (4.1) is defined by the requirement that

$$(a, \mathscr{L}b) = (\mathscr{L}^* a, b), \tag{4.2}$$

for all fields a and b which are $2\pi/\alpha$ periodic in x and have

$$a = b = \partial a/\partial y = \partial b/\partial y = 0 \quad \text{at} \quad y = \pm 1.$$

We find that

$$\mathscr{L}^* a = -\Delta\left(\bar{U}\frac{\partial a}{\partial x}\right) + \bar{U}''\frac{\partial a}{\partial x} - \lambda\Delta^2 a. \tag{4.3}$$

The adjoint eigenvalue problem is thus given by the system

$$-\bar{\gamma}\Delta\psi^* + \mathscr{L}^*\psi^* = 0, \quad \psi^* = \partial\psi^*/\partial y = 0 \quad \text{at} \quad y = \pm 1, \quad \text{periodicity in } x. \tag{4.4a, b, c}$$

5. Perturbation formula for γ'

A glance at the neutral curve of figure 1 shows that, on the arc AC, plane Poiseuille flow loses stability as R is increased past R_c at a fixed value of α. On the arc CD, plane Poiseuille flow gains stability as R is increased past R_c at a fixed value of α. We are going to derive a perturbation formula which gives the value of $d\gamma(\lambda)/d\lambda = \gamma'$ at a fixed α.

From (3.4) we find that

$$(-\gamma\Delta + \mathscr{L})\, d\psi/d\lambda - \gamma'\Delta\psi - \Delta^2\psi = 0, \tag{5.1 a}$$

$$\frac{d\psi}{d\lambda} = \frac{\partial}{\partial y}\left(\frac{d\psi}{d\lambda}\right) = 0 \quad \text{at} \quad y = \pm 1, \quad \text{periodicity in } x. \tag{5.1 b, c}$$

Since

$$-\gamma\left(\Delta\frac{d\psi}{d\lambda}, \psi^*\right) + \left(\mathscr{L}\frac{d\psi}{d\lambda}, \psi^*\right) = -\gamma\left(\frac{d\psi}{d\lambda}, \Delta\psi^*\right) + \left(\frac{d\psi}{d\lambda}, \mathscr{L}^*\psi^*\right) = 0 \tag{5.2}$$

we find that

$$-\gamma'(\Delta\psi, \psi^*) = (\Delta^2\psi, \psi^*). \tag{5.3}$$

This gives

$$\gamma' = (\Delta\psi, \Delta\psi^*)/(\nabla\psi, \nabla\psi^*). \tag{5.4}$$

6. The perturbation series for periodic solutions which bifurcate from plane Poiseuille flow

We return now to the basic problem (2.6). We shall restrict this problem to a search for nonlinear solutions which are periodic in time with period $2\pi/\omega(\epsilon)$. By introducing the Poincaré–Lindstedt frequency mapping

$$\omega(\epsilon)\, t = s,$$

we may reformulate (2.6) as

$$\mathscr{J}\hat{\Psi} + \epsilon J(\hat{\Psi}, \Delta\hat{\Psi}) = 0, \quad \hat{\Psi} = \partial\hat{\Psi}/\partial y = 0 \quad \text{at} \quad y = \pm 1, \tag{6.1 a, b}$$

and

$$\hat{\Psi}(x, y, s) \text{ is periodic with period } 2\pi/\alpha \text{ in } x \text{ and } 2\pi \text{ in } s, \tag{6.1 c}$$

where

$$\mathscr{J}[\omega, \lambda] = \omega\, \partial\Delta/\partial s + \mathscr{L}[\tilde{U}, \lambda]. \tag{6.2}$$

To complete the specification of the problem, we shall need a normalizing condition which will define ϵ. This condition has the form

$$[\Delta\hat{\Psi}] = -k^2, \tag{6.1 d}$$

where k^2 is any fixed positive number to be chosen for convenience. This implies that

$$-\Delta'(\tilde{\tilde{\Psi}} - [\tilde{\Psi}])] = \epsilon k^2. \tag{6.3}$$

The definition of the bracket notation is given in § 7.

The solution of (6.1) can be constructed as a Taylor series

$$\begin{Bmatrix} \hat{\Psi}(x, y, s, \epsilon) \\ \omega(\epsilon) \\ \lambda(\epsilon) \end{Bmatrix} = \sum_{l=0} \epsilon^l \begin{Bmatrix} \Psi_l(x, y, s) \\ \omega_l \\ \lambda_l \end{Bmatrix}, \tag{6.4}$$

where $\lambda_l = (1/l!)\,\partial^l\lambda/\partial\epsilon^l$, etc. Substitution of (6.4) into (6.1) leads us to the following sequence of problems.

At zeroth order

$$\mathscr{J}_0\Psi_0 = 0, \quad \Psi_0 = \partial\Psi_0/\partial y = 0 \quad \text{at} \quad y = \pm 1, \quad [\Delta\Psi_0] = -k^2.\dagger \tag{6.5}$$

At first order

$$\mathscr{J}_0\Psi_1 + \mathscr{J}_1\Psi_0 + J_0 = 0, \quad \Psi_1 = \partial\Psi_1/\partial y = 0 \quad \text{at} \quad y = \pm 1, \quad [\Delta\Psi_1] = 0. \tag{6.6}$$

At second order

$$\left.\begin{aligned} \mathscr{J}_0\Psi_2 + \mathscr{J}_1\Psi_1 + \mathscr{J}_2\Psi_0 + J_1 = 0, \\ \Psi_2 = \partial\Psi_2/\partial y = 0 \quad \text{at} \quad y = \pm 1, \quad [\Delta\Psi_2] = 0. \end{aligned}\right\} \tag{6.7}$$

Here,

$$\mathscr{J}_0 = \mathscr{J}[\omega_0, \lambda_0], \tag{6.8}$$

$$\mathscr{J}_l = \omega_l \frac{\partial}{\partial s}\Delta - \lambda_l \Delta^2 \quad (l > 0), \tag{6.9}$$

$$J_0 = J(\Psi_0, \Delta\Psi_0) \tag{6.10}$$

and

$$J_1 = J(\Psi_1, \Delta\Psi_0) + J(\Psi_0, \Delta\Psi_1). \tag{6.11}$$

The only two solutions of (6.5) which are possible when $\gamma(\lambda_0)$ is a simple eigenvalue of $\mathscr{L}_0$ are

$$z_1 = e^{-is}\psi(x, y; \lambda(\alpha)), \quad z_2 = \bar{z}_1. \tag{6.12}$$

Without loss of generality, we may take the unique real-valued solution of (6.5) in the form

$$\Psi_0 = 2\,\mathrm{Re}\,(z_1). \tag{6.13}$$

The amplitude of this solution is fixed by the normalizing condition (see § 7).

7. The solvability conditions

We shall need to define another scalar product

$$[a, b] = \frac{1}{2\pi}\int_0^{2\pi} (a, b)\, ds \tag{7.1}$$

for 2π periodic functions of s. The bracket

$$[a] \equiv [a, z_1^*], \tag{7.2}$$

where

$$z_1^* = e^{-is}\psi^*, \quad z_1^* = \bar{z}_1^* \tag{7.3}$$

are the solutions of (4.4) at criticality when $\bar{\gamma} = -i\omega_0$. It now follows that (6.3) defines ϵ as the projection of the difference between the periodic solution and Poiseuille flow onto the eigensubspace of the operator $\mathscr{L}_0$.

We may define the adjoint operator

$$\mathscr{J}_0^* = -\omega\,\partial\Delta/\partial s + \mathscr{L}^* \tag{7.4}$$

relative to the scalar product (7.1). The problem adjoint to (6.5) is described by the system

$$\mathscr{J}_0^*\Psi_0^* = 0, \quad \Psi_0^* = \partial\Psi_0^*/\partial y = 0 \quad \text{at} \quad y = \pm 1. \tag{7.5}$$

† We can always choose an arbitrary scale for ϕ_0. To compare our results with those previously given, we chose $\phi_0 = \phi_0^* = 1$ at $y = 0$ This fixes a value for k^2. The results presented in table 1 are based on this normalizing condition.

Suppose that $\pm i\omega_0$ are simple eigenvalues of $\mathscr{L}_0$ and consider

$$\mathscr{J}_0\phi = f, \tag{7.6}$$

where f is a periodic function of x and s with periods $2\pi/\alpha$ and 2π, respectively; $\phi = \partial\phi/\partial y = 0$ at $y = \pm 1$ and ϕ is to have the same periodicity as f in x and s. Thus (7.6) *has solutions if and only if*

$$[f, z_1^*] = [f, z_2^*] = 0. \tag{7.7}$$

Proof. Let $i = 1, 2$ and note that

$$[f, z_i^*] = [\mathscr{J}_0\phi, z_i^*] = [\phi, \mathscr{J}_0^* z_i^*] = 0. \tag{7.8}$$

This proves that (7.7) is a necessary condition for solvability. The proof that it is also sufficient is given in JS.

If f is real valued, then

$$\overline{[f, z_1^*]} = [f, \bar{z}_1^*] = [f, z_2^*]. \tag{7.9}$$

Hence, for real-valued f, the single condition

$$[f] = 0 \tag{7.10}$$

implies both conditions (7.7).

The periodic solutions of (7.6) are not unique. Any solution of the homogeneous problem (6.5) may be added to a solution of (7.6). The condition

$$[\Delta\phi] = 0 \tag{7.11}$$

is sufficient to ensure uniqueness; it guarantees that ϕ is orthogonal to solutions of the homogeneous problem. Our choice of the definition of ϵ is motivated by our desire to generate this convenient orthogonality condition.

On applying the condition (7.10) to (6.6) and (6.7), we find that

$$[\mathscr{J}_1\Psi_0] + [J_0] = 0 \tag{7.12}$$

and
$$[\mathscr{J}_1\Psi_1] + [\mathscr{J}_2\Psi_0] + [J_1] = 0. \tag{7.13}$$

We note that J_0 has no terms proportional to $e^{\pm is}$ and $[J_0] = 0$ by virtue of the integration over the variable s. Hence, at first order,

$$\begin{aligned} 0 = [\mathscr{J}_1\Psi_0] &= \omega_1\left[\frac{\partial}{\partial s}\Delta\Psi_0\right] - \lambda_1[\Delta^2\Psi_0] \\ &= (i\omega_1 - \gamma'\lambda_1)(\nabla\psi_0, \nabla\psi_0^*), \end{aligned} \tag{7.14}$$

where the last equality follows from an integration by parts using (5.4) evaluated at criticality with

$$\psi(x, y, \lambda_0(\alpha)) = \psi_0. \tag{7.15}$$

It follows that

$$\omega_1 = \lambda_1 = \mathscr{J}_1 = 0, \tag{7.16}$$

when $\gamma' \neq 0$.

Turning now to (7.13), we find with use of (7.16) that

$$(i\omega_2 - \gamma'\lambda_2)(\nabla\psi_0, \nabla\psi_0^*) + [J_1] = 0. \tag{7.17}$$

To compute λ_2 and ω_2 we must solve problems (6.5) and (6.6). It is shown in JS that

$$\lambda_{2l+1} = \omega_{2l+1} = 0, \tag{7.18}$$

where l is any integer or zero. This result holds generally and not just for plane Poiseuille flow. Therefore, the first non-zero correction

$$\lambda = \lambda_0 + \lambda_2 \epsilon^2 + O(\epsilon^4) \tag{7.19}$$

and

$$\omega = \omega_0 + \omega_2 \epsilon^2 + O(\epsilon^4) \tag{7.20}$$

can be determined from (7.17) once $[J_1]$ is known. To compute J_1, we must solve the problem (6.6).

8. The solution of the first-order equation

With $\mathscr{J}_1 = 0$ we may write (6.6) as

$$\mathscr{J}_0 \Psi_1 + J_0 = 0, \quad \Psi_1 = \partial\Psi_1/\partial y = 0 \quad \text{at} \quad y = \pm 1, \quad [\Delta\Psi_1] = 0. \tag{8.1a,b,c}$$

Setting

$$\psi_0(x,y) = e^{i\alpha x}\phi_0(y), \quad \psi_0^*(x,y) = e^{i\alpha x}\phi_0^*(y), \tag{8.2}$$

we find that

$$J_0 = J(\Psi_0, \Delta\Psi_0) = A\,e^{2i\theta} + \bar{A}\,e^{-2i\theta} + B + \bar{B}, \tag{8.3}$$

where

$$\theta = \alpha x - s, \quad \Psi_0 = \Psi_0(\theta, y), \tag{8.4}$$

$$A = i\alpha(\phi_0'\phi_0'' - \phi_0\phi_0''') \tag{8.5}$$

and

$$B = -i\alpha[\phi_0'(\phi_0'' - \alpha^2\bar{\phi}_0) + \phi_0(\bar{\phi}_0''' - \alpha^2\bar{\phi}_0')]. \tag{8.6}$$

By linearity we may write

$$\Psi_1 = \phi_{11}e^{2i\theta} + \bar{\phi}_{11}e^{-2i\theta} + \phi_{12} + \bar{\phi}_{12}, \tag{8.7}$$

where

$$\mathscr{J}_0(\phi_{11}e^{2i\theta}) + A\,e^{2i\theta} = 0, \quad \phi_{11}(\pm 1) = \phi_{11}'(\pm 1) = 0 \tag{8.8}$$

and

$$\mathscr{J}_0\phi_{12} + B = 0, \quad \phi_{12}(\pm 1) = \phi_{12}'(\pm 1) = 0. \tag{8.9}$$

Working out (8.8) and (8.9) we get

$$\left(\bar{U} - \frac{\omega_0}{\alpha}\right)(\phi_{11}'' - 4\alpha^2\phi_{11}) - \bar{U}''\phi_{11} + \frac{i\lambda_0}{2\alpha}(\phi_{11}^{\text{iv}} - 8\alpha^2\phi'' + 16\alpha^4\phi_{11}) + \frac{A}{2i\alpha} = 0 \tag{8.10}$$

and

$$\lambda_0\phi_{12}^{\text{iv}} = B. \tag{8.11}$$

We note that, on the neutral curve of figure 1, the critical eigenfunction $\phi_0(y) = \phi_0(-y)$ is a symmetric function. In the calculations we replace the given boundary conditions with symmetric boundary conditions

$$\phi_0'(0) = \phi_0'''(0) = \phi_0(1) = \phi_0'(1) = \phi_0'^*(0) = \phi_0^{*'''}(0) = \phi_0^*(1) = \phi_0^{*\prime}(1) = 0. \tag{8.12}$$

Inspection of A in B and (8.10) and (8.11) shows that they are odd functions of y. We may, therefore, seek odd solutions of (8.10) and (8.11) in the half-interval

$$\phi_{1j}(0) = \phi_{1j}''(0) = \phi_{1j}(1) = \phi_{1j}'(1) = 0 \quad (j = 1, 2). \tag{8.13}$$

Equations (8.10), (8.11) and (8.13) were solved numerically over the half-height of the channel by Runge–Kutta integration, using a filtering technique to remove the 'parasitic error'. This technique was first devised by Kaplan (1964) for the solution of the Orr–Sommerfeld equation and has subsequently been applied to the solution of non-homogeneous equations by many other authors (see, for example, Reynolds & Potter 1967).

To begin with, the eigenvalue problem of the zeroth-order equations (6.5) (i.e. the Orr–Sommerfeld system) was first solved, using the symmetrical conditions (8.12), to obtain the eigenvalue $c_r = \omega_0/\alpha$ for a given point $(\alpha, R_c) = (\alpha, 1/\lambda_0)$ on the neutral curve. The corresponding amplitude function ϕ_0 and its derivatives were also computed. This was done for a range of the parameters (α, R_c). The results agree with those of Fu & Joseph (1970) and are illustrated in figure 1. As a check, the eigenvalue c_r was also computed from the adjoint problem (7.5) using the conditions (8.12). It yielded eigenvalues which agreed very well with those obtained from the Orr–Sommerfeld system. The adjoint eigenfunction ϕ_0^* and its derivatives for the same parametric values (α, R_c) were computed as well.

Once the ϕ_0 problem had been solved, the solutions to the non-homogeneous equations for ϕ_{11} and ϕ_{12}, equations (8.10), (8.11) and (8.13), were obtained for given values of the parameters c_r, α and R_c used in the solution of the ϕ_0 problem. These solutions of the non-homogeneous equations were obtained as the sum of a solution to the corresponding homogeneous equation and a particular solution. The amplitude functions ϕ_{11} and ϕ_{12} and their derivatives result directly from the numerical integration. All the calculations were performed on a CDC 6600 digital computer with single-precision arithmetic.

9. The expressions for λ_2 and ω_2

To calculate λ_2 and ω_2 from (7.17), one begins with the evaluation of $[J_1]$. From (6.11) and (2.7), J_1 can be written as

$$J_1 = \frac{\partial \Psi_1}{\partial y}\frac{\partial}{\partial x}\Delta\Psi_0 - \frac{\partial \Psi_1}{\partial x}\frac{\partial}{\partial y}\Delta\Psi_0 + \frac{\partial \Psi_0}{\partial y}\frac{\partial}{\partial x}\Delta\Psi_1 - \frac{\partial \Psi_0}{\partial x}\frac{\partial}{\partial y}\Delta\Psi_1. \tag{9.1}$$

By employing (8.2), one finds that

$$[J_1] = \frac{2\pi}{\alpha}\int_{-1}^{1} H\bar{\phi}_0^*\,dy, \tag{9.2}$$

where

$$\begin{aligned} H = i\alpha[&-\phi_{11}'(\bar{\phi}_0''-\alpha^2\bar{\phi}_0) - 2\phi_{11}(\bar{\phi}_0'''-\alpha^2\bar{\phi}_0') + 2\bar{\phi}_0'(\phi_{11}''-4\alpha^2\phi_{11}) \\ &+\bar{\phi}_0(\phi_{11}'''-4\alpha^2\phi_{11}') + (\phi_{12}'+\bar{\phi}_{12}')(\phi_0''-\alpha^2\phi_0) - \phi_0(\phi_{12}'''+\bar{\phi}_{12}''')]. \end{aligned} \tag{9.3}$$

Equation (9.2) can be further simplified by integration by parts. This yields

$$\begin{aligned} [J_1] = 4\pi i\bigg\{&\int_0^1 \bar{\phi}_0^{*\prime}[-\bar{\phi}_0\phi_{11}'' - \bar{\phi}_0'\phi_{11}' + 2\bar{\phi}_0''\phi_{11} + \phi_0(\phi_{12}''+\bar{\phi}_{12}'') - \phi_0'(\phi_{12}'+\bar{\phi}_{12}')]\,dy \\ &-\alpha^2\int_0^1 \bar{\phi}_0^*[3\bar{\phi}_0\phi_{11}' + 6\bar{\phi}_0'\phi_{11} + \phi_0(\phi_{12}'+\bar{\phi}_{12}')]\,dy\bigg\}. \end{aligned} \tag{9.4}$$

α	R_c	c_r	ξ'	λ_2	ω_2
0·650	22424	0·1656	192·36	−0·043	34·97
0·700	16355	0·1823	158·44	−0·055	39·34
0·750	12461	0·1983	133·01	−0·066	46·48
0·800	9882	0·2136	113·33	−0·070	57·72
0·850	8141	0·2278	97·57	−0·057	75·60
0·900	6965	0·2408	84·43	−0·010	104·75
0·925	6540	0·2467	78·48	0·035	126·23
0·950	6208	0·2522	72·78	0·103	154·94
1·000	5815	0·2612	61·35	0·359	251·69
1·021†	5772†	0·2640	56·10	0·556	323·42
1·050	5890	0·2664	47·41	1·039	501·49
1·075	6314	0·2658	36·30	2·073	901·37
1·090	7024	0·2624	23·65	4·437	1875·20
1·095	7613	0·2591	14·56	8·589	3648·26
1·0964	7947	0·2572	9·67	14·002	5989·47
1·0964	9356	0·2497	−10·03	−17·293	−7704·32
1·095	9895	0·2470	−17·36	−10·735	−4863·71
1·090	11217	0·2410	−35·06	−6·127	−2888·68
1·075	14307	0·2292	−75·62	−3·562	−1818·73
1·050	19360	0·2147	−140·71	−2·394	−1339·35
1·020	26360	0·2005	−228·83	−1·771	−1067·27
1·000	31896	0·1921	−296·93	−1·507	−937·12
0·980	38329	0·1842	−374·35	−1·303	−824·43

† Minimum critical point

TABLE 1. Neutral stability characteristics for plane Poiseuille flow ($c_i = 0$)

With (9.4), the terms λ_2 and ω_2 can be evaluated from (7.17). By separating (7.17) into real and imaginary parts, one finds

$$\lambda_2 = \mathrm{Re}\,\{[J_1]/(\nabla\psi_0, \nabla\psi_0^*)\}/\mathrm{Re}\,\gamma', \tag{9.5}$$

$$\omega_2 = \lambda_2 \mathrm{Im}\,\gamma' - \mathrm{Im}\,\{[J_1]/(\nabla\psi_0, \nabla\psi_0^*)\}, \tag{9.6}$$

where $\gamma' = (\Delta\psi_0, \Delta\psi_0^*)/(\nabla\psi_0, \nabla\psi_0^*)$ at criticality and, when use is made of (4.1) and (8.2),

$$(\nabla\psi_0, \nabla\psi_0^*) = \frac{4\pi}{\alpha}\int_0^1 (\phi_0'\bar{\phi}_0^{*\prime} + \alpha^2\phi_0\bar{\phi}_0^*)\,dy \tag{9.7}$$

and
$$(\Delta\psi_0, \Delta\psi_0^*) = \frac{4\pi}{\alpha}\int_0^1 (\phi_0''\bar{\phi}_0^{*\prime\prime} + 2\alpha^2\phi_0'\bar{\phi}_0^{*\prime} + \alpha^4\phi_0\bar{\phi}_0^*)\,dy. \tag{9.8}$$

With the ϕ_0, ϕ_0^*, ϕ_{11} and ϕ_{12} solutions available, the expressions for $(\nabla\psi_0, \nabla\psi_0^*)$, $(\Delta\psi_0, \Delta\psi_0^*)$ and $[J_1]$ were evaluated numerically. The values of λ_2 and ω_2 were then calculated from (9.5) and (9.6).

The numerical results are listed in table 1. These results were obtained from calculations employing 100 steps over the interval $0 \leqslant y \leqslant 1$, for a step size of 0·01 was found to give sufficiently accurate results.

Inspection of table 1 shows that $\xi' = d\xi/d\lambda$ changes sign at the point C of the neutral curve of figure 1. The values of $[J_1]$ are everywhere finite. Hence, from (9.5) and (9.6) we see that λ_2 and ω_2 are both unbounded at point C of the neutral curve.

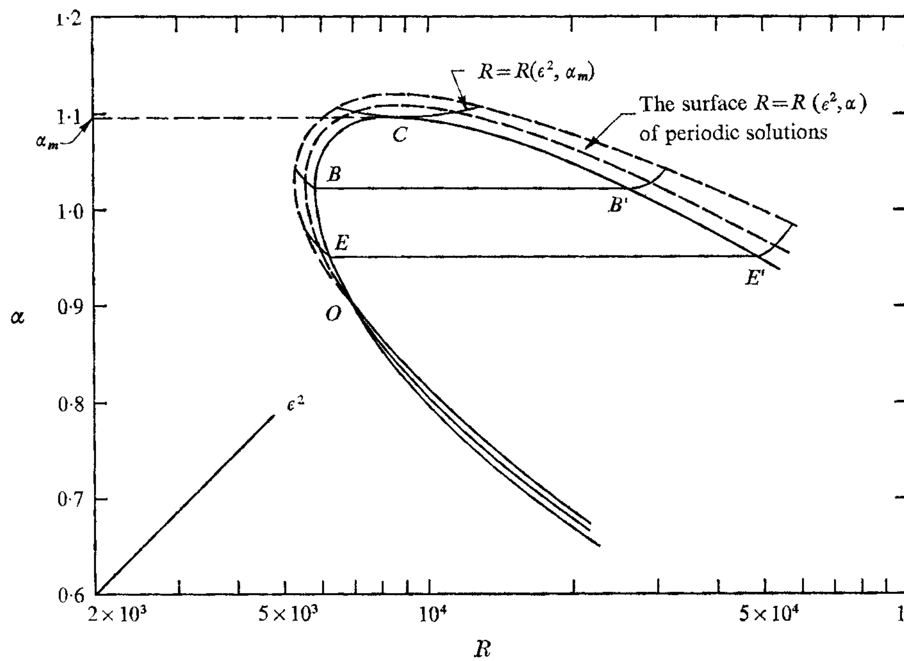

FIGURE 2. Schematic sketch of the surface of periodic solutions which bifurcate from plane Poiseuille flow. The dotted lines indicate subcritical bifurcation.

To understand the results of the numerical computation, we consider the surface

$$R = R(\epsilon^2, \alpha)$$

on which periodic solutions of (6.1) exist. This surface is exhibited in figures 2 and 3. These figures indicate the existence of periodic solutions with

$$R(\epsilon^2, \alpha) = R(0, \alpha_m)$$

and $\alpha > \alpha_m$. This means that the nonlinear problem has periodic solutions with wavelengths shorter than those for the linearized problem.

The numerical results exhibited in table 1 are in qualitative agreement with similar results which have been given by Reynolds & Potter (1967), Pekeris & Shkoller (1967, 1969) and McIntire & Lin (1972).

10. Uniqueness and stability of the solutions

It is a consequence of the bifurcation theory of JS that, if R is a critical value belonging to a simple eigenvalue $\gamma(R) = i\omega(R)$ of the spectral problem, then, apart from phase differences, one and only one periodic solution bifurcates there. Now Squire's theorem ensures that the values $R(\alpha)$ lying on the lower branch AC of the neutral curve of figure 1 are the smallest of the critical values for the full three-dimensional spectral problem. Assuming simplicity, it follows that two-dimensional solutions will bifurcate first from the critical value $R(\alpha)$ on the lower branch of the neutral curve. These 'Tollmien–Schlichting' waves are not observed, however, because they are unstable.

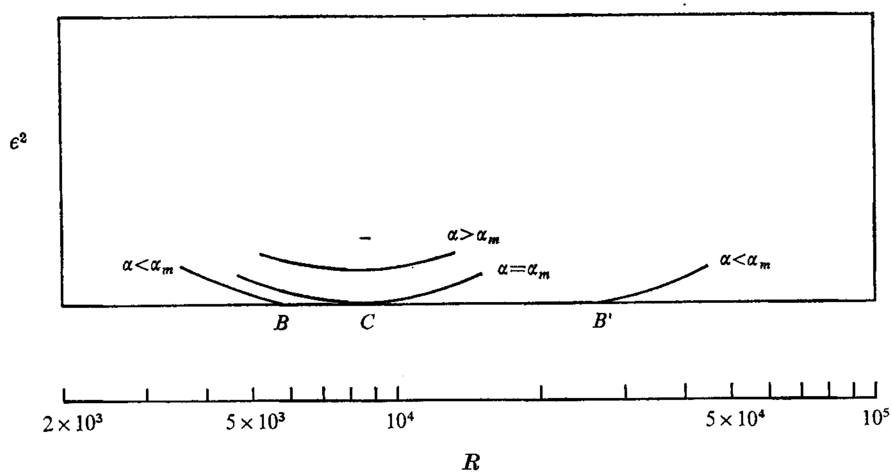

FIGURE 3. Schematic sketch of the projection of the surface of periodic solutions onto the R, ϵ^2 plane. Note that nonlinear solutions with $\alpha > \alpha_m$ are implied by our construction. Therefore, shorter periodic waves are possible in the nonlinear problem than in the linearized problem ($\epsilon^2 = 0$).

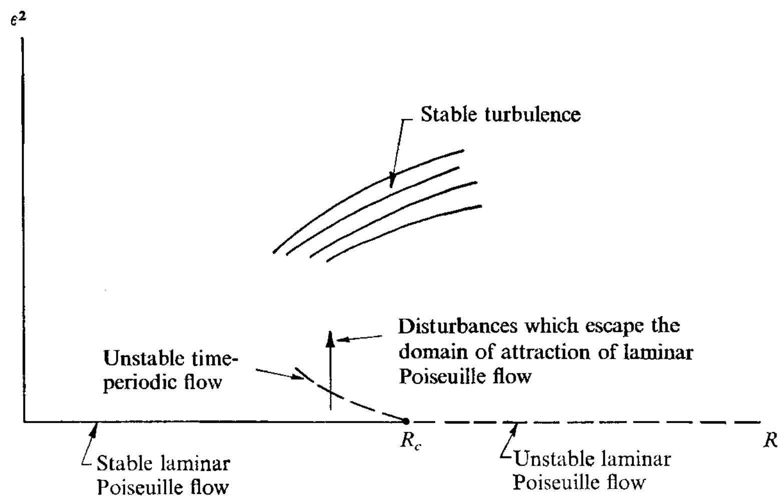

FIGURE 4. Schematic bifurcation diagram for Poiseuille flow. Disturbances which escape the domain of attraction of laminar Poiseuille flow snap through the periodic solution and are attracted by 'turbulent solutions' with larger norms.

We shall omit the proof of stability of supercritical bifurcating solutions (see JS). The main hypothesis of this proof, like that of the bifurcation theory generally, is the simplicity of the eigenvalue γ at criticality. For the two-dimensional problem, this assumption appears to be valid.

The stability of supercritical disturbances is of a very restricted kind. We guarantee stability only to small disturbances having the same wavelength as the bifurcating solution. This last restriction could never be expected to hold in nature and we must reserve judgement about the actual stability of solutions on the lower branch of the neutral curve (below point O on figure 2).

The instability of the subcritical bifurcating solutions is a much more substantial result which is likely not to depend, in any crucial way, on the assumed simplicity of γ. If a solution is unstable to one type of disturbance, it will not be made more stable by allowing more disturbances.

Since most interesting periodic solutions (those in the neighbourhood of point B of the neutral curve) are subcritical, snap-through instabilities of laminar plane Poiseuille flow are to be expected. The discontinuous transitions which are implied by the snap-through instability are in excellent agreement with experimental observations. The conjectured stability picture for plane Poiseuille flow is exhibited in figure 4.

This work was partially supported by grants from the University of Missouri-Rolla (T. S. C.) and the National Science Foundation GK-12500 (D. D. J.).

Appendix. A brief comparison with previous work

The present method differs from the method of amplitude expansions used by Stuart (1960), Reynolds & Potter (1967) and others in several ways. The present method is rigorous; it leads to exact statements about the number of bifurcating solutions, their analytic characterization, their stability properties with respect to Floquet analysis; it is free from arbitrary assumptions; in its general form (given by JS) it applies to steady flow in arbitrary bounded domains;† it leads directly to the nonlinear 'neutral curve', that is, to the locus $R(\epsilon^2)$ of periodic solutions; it has a relatively simple structure which makes the mathematical derivations easy to follow and keeps the notation at a minimum; and it is convenient for numerical computations.

It is difficult to make a direct comparison between the present method and the method of amplitude expansions because the two theories start from different resolutions of the same motion: the method of Stuart and the others starts from horizontally averaged equations and considers fluctuations around this average motion; our method decomposes the motion into a known laminar flow (here, Poiseuille flow (2.4)) plus a disturbance. In parallel flow, where the horizontal averaging can be defined and the method of amplitude expansions should apply, we can make some comparisons at the lowest significant order. The functions Φ, ψ_1, ψ_2 and f of Stuart (1960) correspond to our functions $\overline{\phi}_0^*$, ϕ_0, ϕ_{11} and $\phi_{12}+\overline{\phi}_{12}$. Moreover,

$$-i\omega_2+\gamma'\lambda_2 = \frac{[J_1]}{(\nabla\psi_0, \nabla\psi_0^*)} = i\alpha\frac{\displaystyle\int_0^1 \Phi g\,dz}{\displaystyle\int_0^1 \Phi g_1\,dz} = i\alpha k, \qquad \text{(A 1)}$$

† Readers interested in applying the general theory of JS should make the following corrections: replace ξ with γ in (3.7)–(4.9); ω_m with $\omega_m-\lambda_m\operatorname{Im}\gamma'$ in (4.10), (4.14*b*) and (5.12); σ with $\sigma\omega_0$ in (5.4)–(5.6); the last equation on p. 91 with

$$-\sigma_2\omega_0(1-ia_0)-i\omega_2+\gamma'\lambda_2+3[F_2]_1 = 0;$$

$\epsilon = 0$ with $\epsilon \neq 0$ after (5.6); $\mathbf{u}$ with $\hat{\mathbf{u}}$ in (3.16) and (3.20); $-\overline{\gamma}$ with $\overline{\gamma}$ in (3.2); a_0 with $-a_0$ in (5.9), s with δ in (3.15), and λ_2 with $-\lambda_2$ in (4.14*c*).

(c_i, α, R)	Present work $\gamma'\lambda_2 - i\omega_2$	McIntire & Lin $a^{(2)} + ib^{(2)}$
(0, 1·021, 5772)	$31{\cdot}1 - i173{\cdot}4$	$29{\cdot}67 - i165{\cdot}9$
(0, 1·094, 7500)	$119{\cdot}5 - i308$	$101{\cdot}2 - i258$

TABLE 2. A representative comparison of results

where the second ratio of integrals is equation (4.20) of Stuart (1960) and k is a Landau constant.

Given ω_2, λ_2 and γ' we may compute the complex Landau constant

$$i\alpha k(= -i\omega_2 + \gamma'\lambda_2)$$

of Stuart. On the other hand, computation of the Landau constant k does not fix the value of λ_2 since γ' must also be computed.

The comparison of our numerical results with those of Reynolds & Potter (1967) is facilitated by their 'conversion of notation' (table 1). According to that table their $a^{(2)} + ib^{(2)} = i\alpha k = -i\omega_2 + \gamma'\lambda_2$. McIntire & Lin (1972) use the notation and equations of Reynolds & Potter and in table 2 we have made a representative comparison.

REFERENCES

FU, T. S. & JOSEPH, D. D. 1970 Linear instability of asymmetric flow in channels. *Phys. Fluids*, **13**, 217.

JOSEPH, D. D. 1971 Stability of convection in containers of arbitrary shape. *J. Fluid Mech.* **47**, 257.

JOSEPH, D. D. 1973 Remarks about bifurcation and stability of quasiperiodic solutions which bifurcate from periodic solutions of the Navier–Stokes equations. In *Nonlinear Problems in Physical Science and Biology, Springer Lecture Notes in Mathematics* (ed. I. Stakgold, D. Joseph & D. Sattinger), no. 322.

JOSEPH, D. D. & SATTINGER, D. H. 1972 Bifurcating time periodic solutions and their stability. *Arch. Rat. Mech. Anal.* **45**, 79.

KAPLAN, R. E. 1964 The stability of laminar incompressible boundary layers in the presence of compliant boundaries. *M.I.T., Aeroelastic & Structures Res. Lab. Rep.* ASRL TR 116-1.

MCINTIRE, L. V. & LIN, C. H. 1972 Finite amplitude instability of second-order fluids in plane Poiseuille flow. *J. Fluid Mech.* **52**, 273.

PEKERIS, C. L. & SHKOLLER, B. 1967 Stability of plane Poiseuille flow to periodic disturbances of finite amplitude in the vicinity of the neutral curve. *J. Fluid Mech.* **29**, 31.

PEKERIS, C. L. & SHKOLLER, B. 1969 The neutral curves for periodic perturbations of finite amplitude of plane Poiseuille flow. *J. Fluid Mech.* **39**, 629.

REYNOLDS, W. C. & POTTER, M. C. 1967 Finite-amplitude instability of parallel shear flows. *J. Fluid Mech.* **27**, 465.

STUART, J. T. 1960 On the non-linear mechanics of wave disturbances in stable and unstable parallel flows. Part 1. The basic behaviour in plane Poiseuille flow. *J. Fluid Mech.* **9**, 353.

WATSON, J. 1960 On the non-linear mechanics of wave disturbances in stable and unstable parallel flows. Part 2. The development of a solution for plane Poiseuille flow and for plane Couette flow. *J. Fluid Mech.* **9**, 371.